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DETERMINATION OF THE SPIN AXIS OF A SPINNING SATELLITE

BY

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ABSTRACT

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Many of the satellites sent into orbit during the last few years are spinning for stabilization and or other reasons. The purpose of this paper is to determine the spin axis, a unit vector $\vec{S}^\circ$, of the satellite using ground tracking information. It will be shown that range rate information is adequate for this purpose. The determination of $\vec{S}^\circ$ is necessary either to check the onboard sensors once the spacecraft is in orbit or in case of a slight onboard malfunction, the spin axis could not be determined otherwise.

Author

CONTENTS

	Page
ABSTRACT	iv
INTRODUCTION	1
I. RANGE RATE MODULATION DUE TO SATELLITE SPIN	1
II. DETERMINATION OF THE COMPONENTS OF THE SATELLITE SPIN AXIS UNIT VECTOR $\vec{S}^{\circ}$ AND THEIR ERRORS	5
III. ESTIMATION OF THE MEASUREMENT VARIANCE	11
IV. CONCLUSIONS	12
REFERENCES	13

DETERMINATION OF THE SPIN AXIS OF A SPINNING SATELLITE

INTRODUCTION

On many occasions it is most desirable or even necessary to determine the spin axis of a rotating satellite in space. The reasons for this are for instance check to onboard sensors, or to determine the spin axis in case of a malfunction. In this case many of the carried experiments can still be evaluated as long as the spin axis is known.

One of the necessary conditions to do so with the method outlined here is that the radiating antenna element is displaced from the spin axis center since the Doppler modulation imposed and the range rate by the rotating satellite is utilized for the analysis. The range rate or Doppler as commonly used of a satellite in orbit can be determined with great precision. (See References 1, 2, 3, and 4.) Errors in the order of cm/s can and have been achieved with cw-ranging systems. This fact will be utilized to determine the spin axis (unit vector $\vec{S}^{\circ}$) of a rotating satellite.

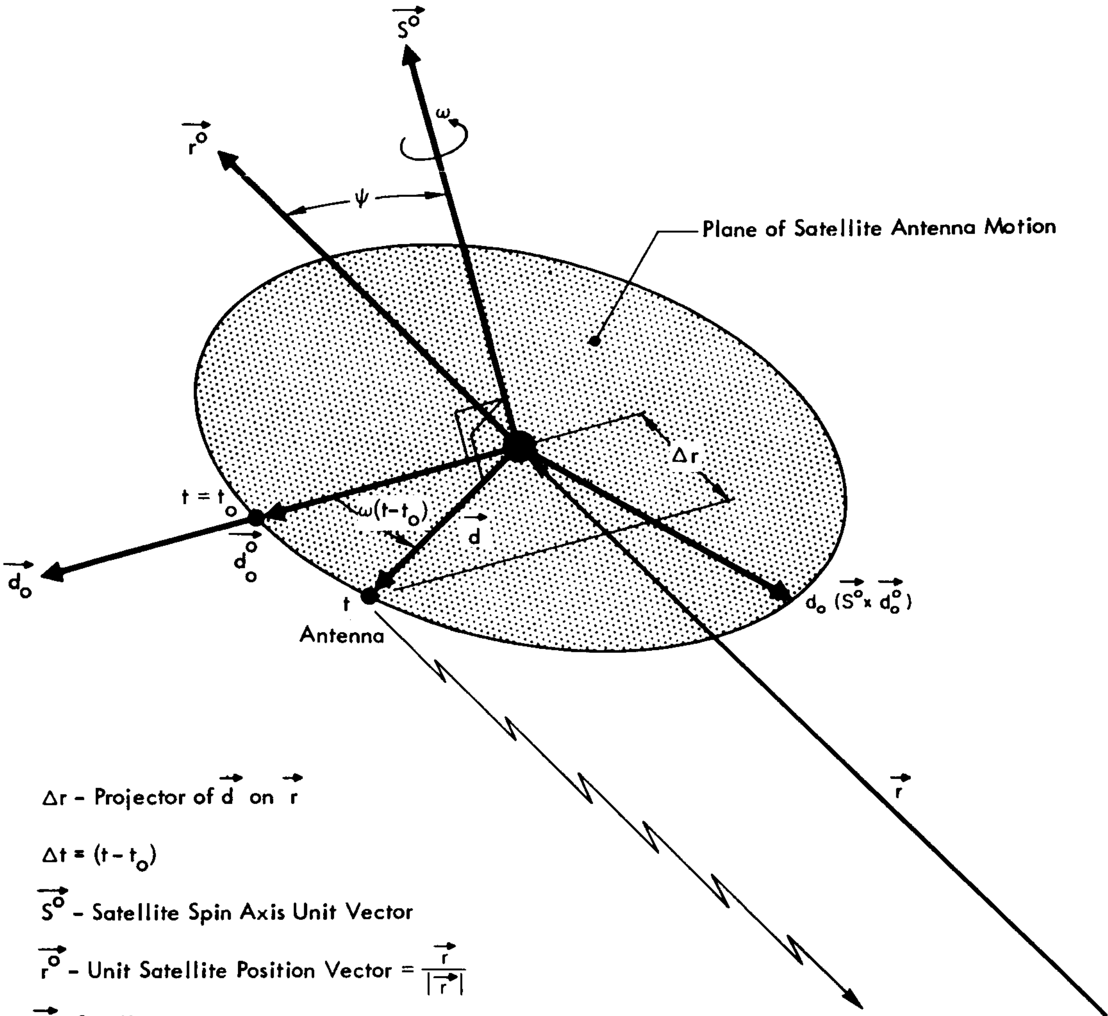
The only assumptions made here are (a) that the antenna of the satellite is displaced from its axis of symmetry, (b) that the spacecraft is spinning only, and (c) that the satellite orbit can or has been determined.

I. RANGE RATE MODULATION DUE TO SATELLITE SPIN

Some of our satellites are spinning in space and have their radiating element, the antenna, displaced from its spin axis.*

It will now be shown in this paper how the range rate $\dot{r}$ is modulated by the spinning motion of the spacecraft. Figure 1 and 1a depict the vector diagram of the antenna motion in detail as well as the angle ψ = arc cos $(\vec{r}^{\circ}, \ \vec{s}^{\circ})$ between the satellite spin axis $\vec{s}^{\circ}$ and the satellite position vector $\vec{r}$ (of $\vec{r}^{\circ}$). This angle, as shown later, will be used for the determination of $\vec{s}^{\circ}$. Using vector

*See reference 5 for centered turn stile antenna.



Δr - Projector of $\vec{d}$ on $\vec{r}$

$\Delta t = (t - t_o)$

$\vec{S^o}$ - Satellite Spin Axis Unit Vector

$\vec{r^o}$ - Unit Satellite Position Vector $= \frac{\vec{r}}{|\vec{r}|}$

$\vec{r}$ - Satellite Position Vector

$\psi = \text{arc cos } (\vec{r^o} \cdot \vec{S^o})$

ω - Satellite Rotation

$\vec{d_s}$ - Starting Antenna Position Vector

$\vec{d}$ - Antenna Position Vector

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Figure 1—Antenna Motion Vector Diagram

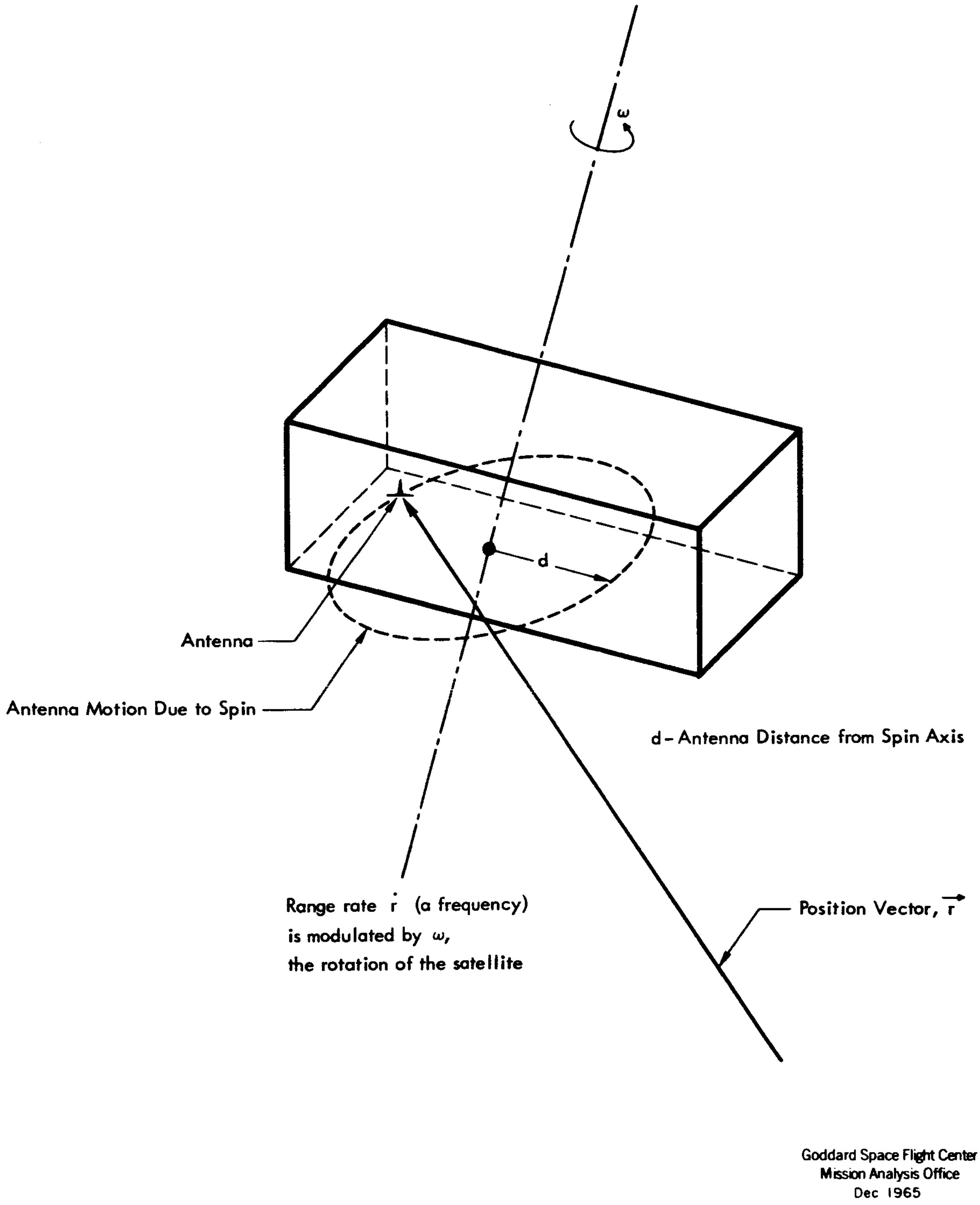


Figure 1a—Satellite and Antenna Details

notation, the antenna position vector $\vec{d}$ can be written as

$$\vec{d} = \vec{d}_0 \cos \omega\Delta t + d(\vec{s}^\circ \times \vec{d}_0^\circ) \sin \omega\Delta t \tag{1}$$

where $\vec{d}^\circ = d \cdot \vec{d}_0^\circ$ * is the starting position of the spacecraft antenna at $t = t_0$ or $\Delta t = 0$. Equation 1 follows directly from Figure 1 as can be seen. Projecting $\vec{d}$ onto the range vector $\vec{r}$ results in:

$$\Delta r = (\vec{d} \cdot \vec{r}^\circ) = \frac{1}{r} (\vec{d} \cdot \vec{r}) \tag{2}$$

Since, as assumed, the spacecraft is spinning, the value Δr changes with time (also due to orbital motion which is assumed to be negligible as is shown later), that is:

$$\Delta \dot{r} = \frac{1}{r^2} \left\{ r \left[(\dot{\vec{d}} \cdot \vec{r}) + (\vec{d} \cdot \dot{\vec{r}}) \right] - (\vec{d} \cdot \vec{r})\, \dot{r} \right\} \tag{3}$$

which is obtained by simply differentiating (2) with respect to time.

Differentiating (1) also with respect to time and introducing it into (3) yields after some manipulation:

$$\Delta \dot{r} = -\omega d(\vec{d}_0^\circ \cdot \vec{r}^\circ) \sin \omega\Delta t + \omega d(\vec{d}_0^\circ \cdot (\vec{r}^\circ \times \vec{s}^\circ)) \cos \omega\Delta t +$$

$$+ \left(\frac{d}{r}\right) (\vec{d}^\circ \cdot \dot{\vec{r}}) - \left(\frac{d}{r}\right) \dot{r}\, (\vec{d}^\circ \cdot \vec{r}^\circ) \tag{4}$$

*A unit vector will always be designated with a zero superscript, that is $\vec{x}^\circ = \vec{x}/|\vec{x}|$.

Inspecting (4) shows that as long as

$$\frac{|\dot{\vec{r}}|}{r} << \omega \quad \text{and} \quad \frac{\dot{r}}{r} << \omega \tag{5}$$

the third and the fourth term of (4) can be neglected.*

Since $\vec{d}^{\circ}_{0}$ is an arbitrary starting vector (unit vector) it will be chosen so that

$$(\vec{d}^{\circ}_{0} \cdot \vec{r}^{\circ}) = 0 \tag{6}$$

or

$$\vec{d}^{\circ}_{0} = \frac{1}{\sin \psi} (\vec{r}^{\circ} \times \vec{s}^{\circ})$$

Introducing (6) into (4) neglecting the terms just mentioned yields:

$$\Delta \dot{r} = (\omega d \sin \psi) \cos \omega \Delta t \tag{7}$$

with

$$\frac{v}{r} << \omega, \quad \frac{\dot{r}}{r} << \omega$$

as the "Range Rate Modulation" due to the spinning motion of a satellite.

---

*Example: $|\dot{\vec{r}}| \doteq |\vec{v}|$ (neglect earth rotation) $\doteq 8000$ m/s
$n = 9$ revolutions per min., $\omega = \pi n/30 = 1 \text{ sec}^{-1}$
$r \doteq 400$ km (slant range)
than: $|\dot{\vec{r}}|/r \doteq 2.10^{-2} \text{ sec}^{-1} << 1 \text{ sec}^{-1}$

If the range rate modulation $\Delta\dot{r}_i$ from the i^{th} ground tracking station, is given as shown in Figure 2. The angle ψ_i between the spin axis and the position vector $\vec{r}_i$ (or $\vec{r}_i^{\,o}$) can be determined from (7) that is

$$\psi_i = \arc\sin\left(\frac{\Delta\dot{r}_{i\,max}}{\omega_i d}\right) \tag{8}$$

since $\omega = 2\pi/T$ is given by the same process. A better determination of ω can be obtained from the power spectrum of the range rate information as shown in Figure 3.

This angle ψ_i will now be used to determine the components of the unit vector $\vec{s}^{\,o}$, the satellite spin axis.

II. DETERMINATION OF THE COMPONENTS OF THE SATELLITE SPIN AXIS UNIT VECTOR $\vec{s}^{\,o}$ AND THEIR ERRORS

The angle ψ_i is certainly related to the satellite spin axis as can be seen from Figure 4. In reality, the mathematical relation can simply be expressed as a vector dot product, that is

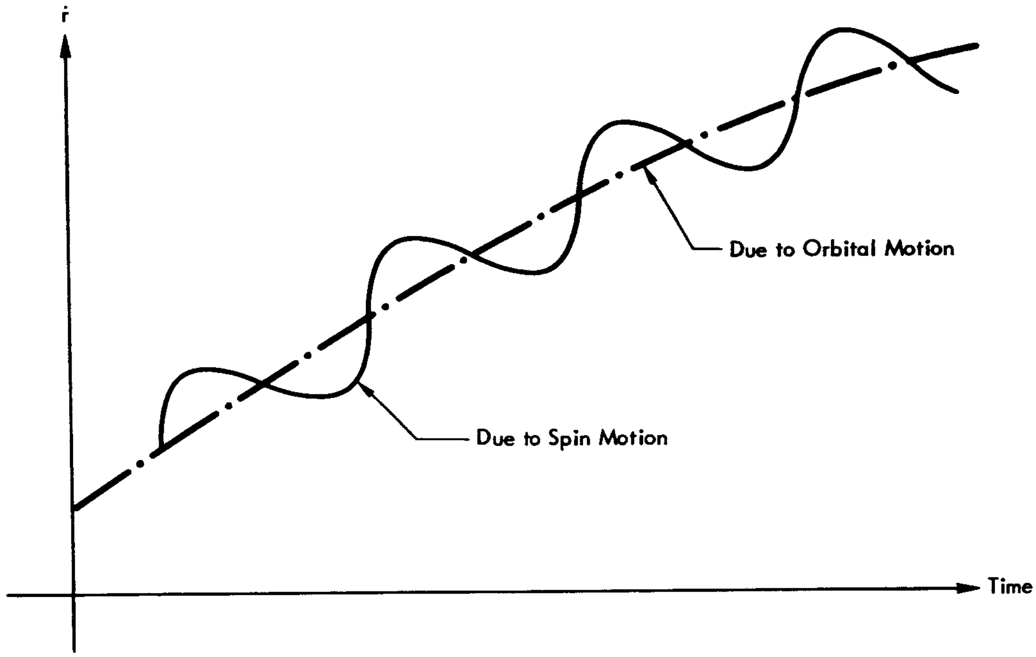
$$\cos\psi_i = (\vec{r}_i^{\,o}\cdot\vec{s}^{\,o}) \tag{9}$$

where $\vec{r}_i^{\,o}$ is the unit satellite position vector of the i^{th} ground station. From (8), the angle ψ_i can be determined, thus only $\vec{s}^{\,o}$ is unknown in equation (9). A straight forward solution of (9) would be to measure from 3 stations the $\Delta\dot{r}_i$ and determine the components (s_1, s_2, s_3) from (9).

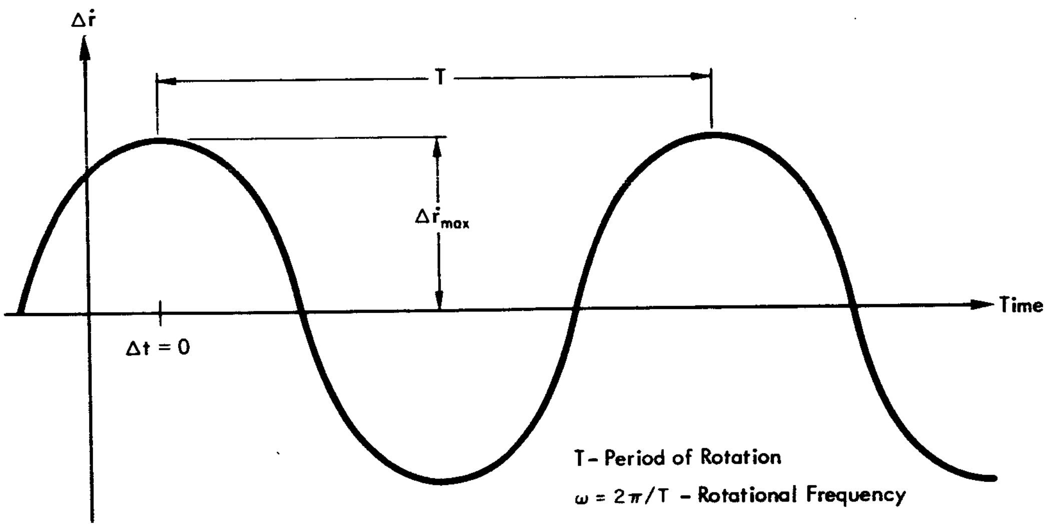
A much better solution can be obtained however by utilization of all information which is usually available, that is using much more than three measurements. In practice many of our ground stations take a large number (several hundreds) of range rate measurements. This in turn dictated a least square solution since more than the necessary three equations (measurements) are thus available.

The dot product of (9) can also be written in component form as:

$$\cos\psi_i = r_{i1}\, s_1 + r_{i2}\, s_2 + r_{i3}\, s_3 \tag{10}$$



(a) "Spin Modulated" Range Rate



(b) Range Rate Modulation (Orbital Motion Subtracted)

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Figure 2

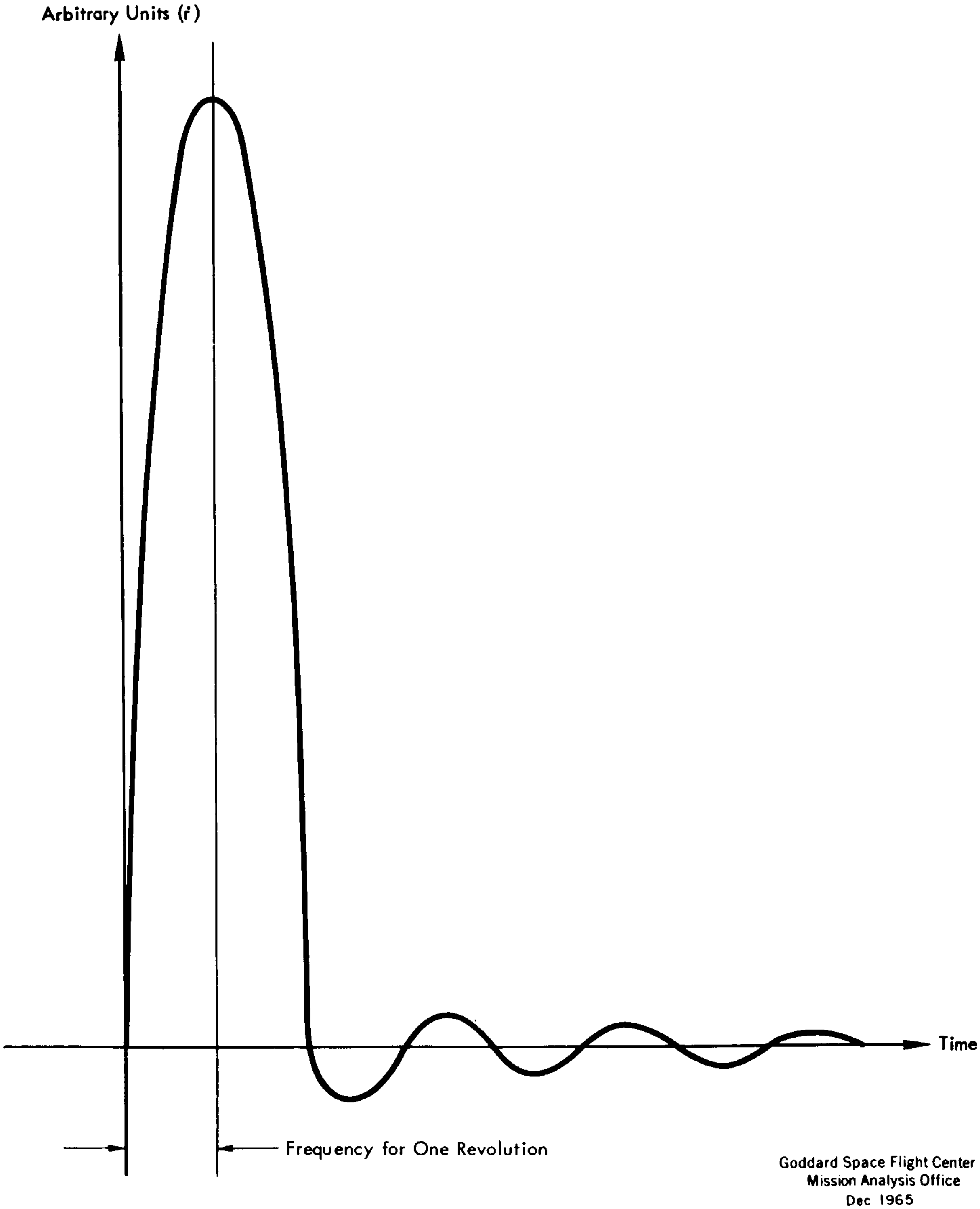

Figure 3–Power Density Spectrum

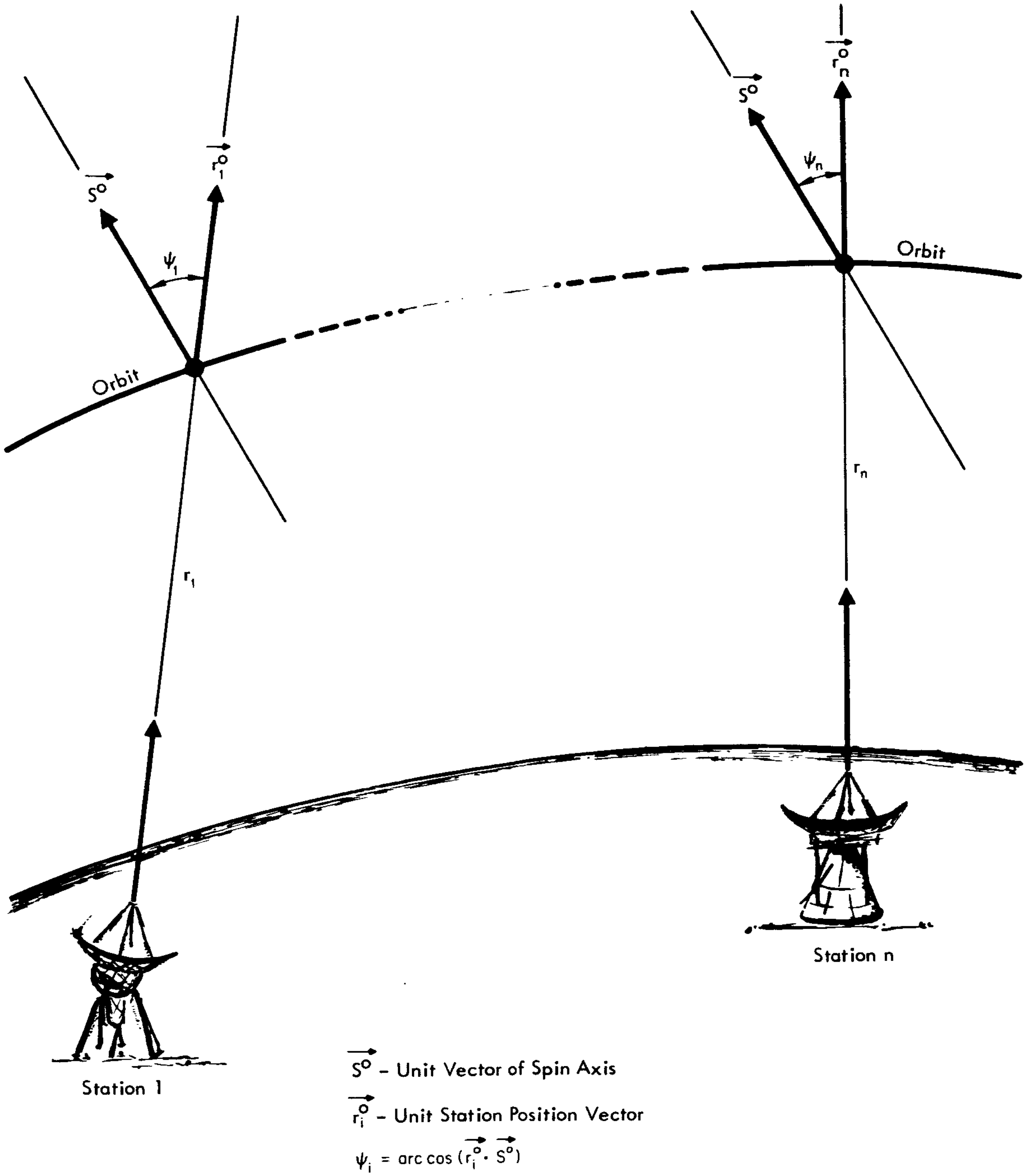

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Figure 4

where of course the values r_{i1}, r_{i2}, r_{i3}, and s_1, s_2, s_3, are the components of the unit vectors $\vec{r}_i^{\circ}$ and $\vec{s}^{\circ}$ respectively.

Using as mentioned many measurements of $\Delta\dot{r}_i$ and ω_i from which ψ_i can be calculated using (8) one has a set of linear equations at hand. From (10) one obtains for n measurements of $\Delta\dot{r}_i$ and ψ_i the following:

$$
\begin{aligned}
a_1 &= \cos\psi_1 = r_{11}\, s_1 + r_{12}\, s_2 + r_{13}\, s_3 \\
a_2 &= \cos\psi_2 = r_{21}\, s_1 + r_{22}\, s_2 + r_{23}\, s_3 \\
&\;\;\vdots \\
a_n &= \cos\psi_n = r_{n1}\, s_1 + r_{n2}\, s_2 + r_{n3}\, s_3
\end{aligned}
\tag{11}
$$

Equation (11) suggested by itself a matrix treatment which will be followed from now on. In matrix form (11) reads simply

$$A_{(n\times 1)} = R_{(n\times 3)} \cdot S_{(3\times 1)} \tag{12}$$

where the matrices A, R and S (the unit vector of the satellite spin axis) are as follows:

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_n \end{bmatrix}_{(A\times 1)}, \quad R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ r_{n1} & r_{n2} & r_{n3} \end{bmatrix}_{(n\times 3)}$$

$$s = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}_{(3\times 1)}$$

In order to solve (12) in the "least square" sense, standard procedures are being applied. (See References 6, 7, 8, 9, 10, 11, and 12.)

Since in practice (12) as such can and will never exist due to inhanced errors in the measurements as well as in the analyses one has to write instead:

$$A_{(n\times 1)} = R_{(n\times 3)} \cdot S_{(3\times 1)} + \epsilon_{(n\times 1)} \tag{13}$$

where ϵ is an error matrix with the following properties; the expectation

$$E(\epsilon) = 0 \tag{14}$$

and the dispersion matrix

$$V(s)_{(n\times n)} = E(\epsilon\epsilon^T) = \sigma^2_{ai}\ I_{(n\times n)} \tag{15}$$

where $I_{(n\times n)}$ is a unit matrix and σ^2_{ai} is the variance of the measurements made. This means in view of (11) to each equation in (11) an error term has to be added to encompass reality.

Equation (13) is that equation which has to be used to determine the matrix S or the unit vector $\vec{s}^{\,o}_i$. The principle of "least squares" requires to solve for S in such a way that the sum of the errors squared ($\epsilon^T\epsilon$) is a minimum. (ϵ^T is the transpose of ϵ.)

A necessary condition for this is that

$$\frac{\partial}{\partial S}(\epsilon^T\epsilon) = 0 \tag{16}$$

or using (13) one obtains from (16)

$$\frac{\partial}{\partial S}\left[(A^T - S^T R^T)\cdot(A - RS)\right] = 0$$

which yields after some calculations the "best" estimate for S, that is

$$\hat{S}_{(3\times 1)} = (R^T R)^{-1}_{(3\times 3)}\ R^T_{(3\times n)}\ A_{(n\times 1)} \tag{17}$$

Equation (17) represents the solution to the problem of the determination of the satellites spin axis.

In order to estimate the errors made in the determination of $\vec{s}^{\circ}$, its dispersion matrix has to be calculated. From (17) and (15) one obtains:

$$\hat{S} = (R^T R)^{-1} R^T (RS + \epsilon) \tag{18}$$

Using (15) and introducing (18) one obtains after some calculations,

$$V(\hat{S}) = E(\epsilon\epsilon^T) = \sigma^2_{ai} (R^T R)^{-1} \tag{19}$$

for the dispersion matrix of $\hat{S}$.

III. ESTIMATION OF THE MEASUREMENT VARIANCE

In order to evaluate the dispersion matrix $V(\hat{S})_{(3\times 1)}$ the measurement variance has also to be estimated.

From (9) it is to be remembered that $\Delta\dot{r}_1$ and ω_i are "measured" quantities which in turn have errors associated with them. These can be estimated from the noise of the data and this is to be assumed here as known. From equations (8) and (11) one can write

$$\sin\psi_i = \frac{1}{d}\cdot\frac{\Delta\dot{r}_{i\,max}}{\omega_i} \tag{20}$$

and

$$\cos\psi_i = a_i$$

Varying both equations, one obtains for the variation in a_i

$$\delta a_i = \frac{\Delta\dot{r}_{i\,max}}{(d\omega_i)^2\sqrt{1-\left(\frac{\Delta\dot{r}_{i\,max}}{d\omega_i}\right)^2}}\left[\delta(\Delta\dot{r}_{i\,max}) - \Delta\dot{r}_{i\,max}\left(\frac{\delta\omega_i}{\omega_i}\right)\right] \tag{21}$$

Assume now that the errors are uncorrelated and replace the value $\Delta\dot{r}_{i\,max}$ by $(\Delta\dot{r}_{i\,max} + \sigma\Delta\dot{r}_{i\,max})$ since it can never be zero due to the inherent errors in its determination, one obtains from (21):

$$\sigma_{ai} = \frac{\Delta\dot{r}_{i\,max} + \sigma\Delta\dot{r}_{i\,max}}{(\omega_i d)^2 \sqrt{1 - \left(\dfrac{\Delta\dot{r}_{i\,max} + \sigma\Delta\dot{r}_{i\,max}}{\omega_i d}\right)^2}} \left[\sigma\Delta\dot{r}^2_{i\,max} + \Delta\dot{r}^2_{i\,max}\left(\frac{\sigma\omega_i}{\omega_i}\right)^2\right]^{1/2} \qquad (22)$$

With the "Measurement" variance σ^2_{ai} and the matrix R known the disposition matrix $V(\hat{S})$ given by (19) can be evaluated.

Equation (22) shows also the limitations for σ_{ai}; $(\Delta\dot{r}_{i\,max} + \sigma\Delta\dot{r}_{i\,max}/\omega_i d)$ has to be smaller than 1 to prevent too large errors.

IV. CONCLUSIONS

It has been shown how the range rate modulation of a spinning satellite with a displaced antenna can be used to determine its spin axis (unit vector) in space. An estimate of the spin axis errors associated with this method is also presented.

It is interesting to note that no additional equipment on the satellite as well as on the ground is necessary. This method may well be suited for spin axis determination of communications and navigation satellites. It further could be used to check the axis of spin stabilized kick stages if deemed necessary by adding a Doppler transponder and antenna to the stage.

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